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Critical exponents for the long-range Ising chain using a transfer matrix approach

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Abstract. The critical behavior of the Ising chain with long-range ferromagnetic interactions decaying with distance r^α, $1 < \alpha < 2$, is investigated using a numerically efficient transfer matrix (TM) method. Finite size approximations to the infinite chain are considered, in which both the number of spins and the number of interaction constants can be independently increased. Systems with interactions between spins up to 18 sites apart and up to 2500 spins in the chain are considered. We obtain data for the critical exponents ν associated with the correlation length based on the Finite Range Scaling (FRS) hypothesis. FRS expressions require the evaluation of derivatives of the thermodynamical properties, which are calculated with the help of analytical recurrence expressions obtained within the TM framework. The Van den Broeck extrapolation procedure is applied in order to estimate the convergence of the exponents. The TM procedure reduces the dimension of the matrices and circumvents several numerical matrix operations.

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1 Introduction

It has been well-known that the thermodynamical properties and critical behavior of physical models is affected by the presence of long-range interactions. They define completely new classes of models, where cooperative effects are enhanced and change the thermodynamical properties in comparison to those of the short-range interaction models. Due to the presence of simultaneous coupling among many degrees of freedom, the long-range models offer very difficult technical problems that makes it impossible to derive exact expressions for the thermodynamical properties within the equilibrium statistical ensemble formulation. This is observed already for the most simple long-range Ising chain, which has been investigated for many decades. In its most simple version, each spin σ_i interacts with all other spins on the chain mediated by coupling constants $J_r = J/r^\alpha$, where r is the distance between the interacting spins measured in integer number of lattice spacings.

Although no closed form solution for this model is available, there are several rigorous results on the existence of distinct thermodynamical phases, what depends on the range of values of α. The most important features of these rigorous results are: for $\alpha > 2$, the system shows only a disordered phase, $\forall T$ [1,2]; for $1 < \alpha \leq 2$, there is a phase transition at finite temperature T_c [3,4]; critical mean field behavior occurs for $1 < \alpha \leq 1.5$ [1,2]; for $\alpha < 1$, only one ordered phase exists, $\forall T$. Regarding the

evaluation of approximate results, both renormalization group schemes [5–8] and numerical calculations of finite size systems [7–10] have been used to estimate the thermodynamical properties, the critical temperature and the critical exponents when $1 < \alpha \leq 2$.

More recently, the scaling behavior of this model, specially in the range $0 \leq \alpha \leq 1$, where the energy is not an extensive quantity, has also been addressed. Investigations have been motivated by a universal scaling scheme proposed by Tsallis [11–13], that should be valid for both extensive and non-extensive long-range models, as those constituted by spins [14–17] rotors [18,19], and so on.

In this work we use a numerically efficient transfer matrix (TM) approach [20] to analyse the critical behavior of the system, estimating the critical exponents ν associated with the correlation length in the range for $1 < \alpha \leq 2$, where the system undergoes a phase transition. In a previous paper [21], we have introduced this approach to check the validity of Tsallis' scaling conjecture to the Ising long-range chain, and to find estimates for the critical temperature. We obtained results that show very good accordance with other numerical estimates, indicating that the proposed approach is quite reliable. It takes into account the long-range interactions between spins, up to a certain distance g apart, in the evaluation of the thermodynamical properties of the system. Also, this method allows to independently increase the number of spins N in the chain, so that $N \geq g + 1$.

The TM approach used herein makes use of very compact matrices, so that the configuration energies and
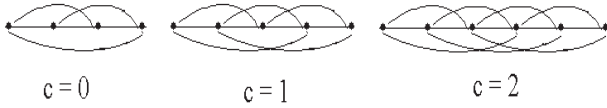
Fig. 1. A schematic representation of the construction of the infinite 3-chain ($g = 3$) for $c = 0, 1$ and 2.

Boltzmann weights, that are numerically evaluated, can be stored and operated in a very efficient way. For instance, we note that the TM procedure avoids the numerical evaluation of the TM eigenvalues. For the evaluation of ν, this framework is quite useful, as it leads to analytical expressions for the derivatives of the Boltzmann weights that can be similarly stored in very compact matrices. So, the derivatives of thermodynamical properties can be directly computed, avoiding the use of numerical differentiation [7].

This work is organized as follows: in Section 2 we discuss the essential aspects of transfer matrix method (TM) used to evaluate the critical exponents. In Section 3 we present the critical temperature, the critical exponent ν associated with the correlation length for values of α in the following range: $1 < \alpha < 2$. In Section 4 we discuss the results and compare with some previous results [7]. Section 5 closes the work with our final remarks and perspectives.

2 Transfer matrix framework

The long-range Ising chain is described by the Hamiltonian

$$\mathcal{H} = -\sum_{i=0}^{\infty} \sum_{r=1}^{\infty} J_r \sigma_i \sigma_{i+r} - h \sum_{i=0}^{\infty} \sigma_i, \qquad (1)$$

where $\sigma_i = \pm 1$ is a classical spin at site i, h is an external field and J_r are the coupling interactions between spins at distance r.

One can construct this model by starting with an infinite chain but a finite number interaction constants. Then, in a series of steps g, all J_g are introduced at once, among all pairs of spins that are g sites apart. It is also possible to construct the model by taking a finite chain with g distinct coupling constants and $N = g + c + 1 \geq g + 1$ spins. Thus, for a generic value g, if c new spins are included in the chain, $g \times c$ new interactions, mediated by J_r, $r = 1, 2, \dots g$, are added, linking every new spin to all spins up to g sites apart that have been introduced at the previous generations. In Figure 1, we show, for example, for fixed $g = 3$, the interactions for the first 3 values of c. If $g \to \infty$, $N \to \infty$, so that this procedures leads to the same actual system. However, for finite values of g, the first construction procedure can be recovered by letting $c \to \infty$.

In this work, we consider the second way of construction quoted above, which is schematically represented in Figure 1, for which the finite chain has $N = g + c + 1$ spins that interact with all spins up to g sites apart. This way, it is possible to obtain a partition function based on 2×2 TM

$M_{g,c}$, instead of $2^g \times 2^g$ TM $\widehat{M}_g$ introduced in reference [7]. Besides working with smaller TM's, this approach has the advantage of avoiding the use of numerical diagonalization procedures to evaluate the largest eigenvalues. According to this TM scheme, which has been described with enough details in [21], the 2×2 TM $M_{g,c}$ are given by:

$$M_{g,c} = \left(\prod_{k=1}^{g} P_k\right) (Q_g \cdot P_g)^c \cdot L_g \equiv R_{g,c} L_g. \qquad (2)$$

where L_g is a $2^{g+1} \times 2$ matrix whose elements $(L_g)_{i,j} = 1$, for $i + j$ even, and $(L_g)_{i,j} = 0$, for $i + j$ odd; P_k are recursively expressed by:

$$(P_k)_{i,j} = \begin{cases} (P_{k-1})_{i,j} \, a_k^{(-1)^{j-1}} & \text{for } i \leq 2^{k-1}, \\ & \text{and } j \leq 2^k \\ (P_k)_{2^k-i+1,2^{k+1}-j+1} & \text{for } 2^{k-1} \leq i \leq 2^k, \\ & \text{and } 2^k \leq j \leq 2^{k+1} \\ 0 & \text{otherwise} \end{cases}$$
$$(3)$$

where $a_k = exp(J_k/T)$; Q_g is a $2^{g+1} \times 2^g$ matrix defined by

$$(Q_g)_{i,j} = \begin{cases} 1 & \text{for } i = j \text{ or } i = j + 2^g \\ 0 & \text{otherwise.} \end{cases} \qquad (4)$$

Therefore the free energy per spin, $f_{g,c} = -T \ln(Z_{g,c})/N$ follows from the partition function

$$Z_{g,c} = 2\lambda_{g,c}^+ = 2((M_{g,c})_{1,1} + (M_{g,c})_{1,2}) = \sum_{i,j}(R_{g,c})_{i,j}. \qquad (5)$$

The correlation function between the first and the r-th spins along the chain, restricted to the case $c = 0$, defined by a g-dependent correlation function $C_g(r;T) = \langle \sigma_1 \sigma_r \rangle_g$, $r = 1, \dots, g$, is given by

$$C_g(r;T) \equiv \frac{1}{Z_g} \sum_{i,j} \left[\underline{P}_1 \left(\prod_{k=2}^{g} P_k\right) L_{g,r} \right]_{i,j}, \qquad (6)$$

where

$$\underline{P}_1 = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ 0 & 0 & -b_1 & -a_1 \end{pmatrix}, \qquad (L_{g,r})_{i,j} = -1^{q_{g,r}(j)},$$

$$q_{g,r}(j) = L\left[\frac{j-1}{2^{g-r}}\right], \qquad (7)$$

with $b_1 = a_1^{-1}$ and $L[x] \equiv$ largest integer in x.

The definition of $C_g(r;T)$ can be extended to $C_{g,c}(i, i+r;T)$, where $c > 0$, $r > g$. If we use (6) for $r = g$, we have $C_g(g;T) = \lambda_g^-/\lambda_g^+$. Note further that the correlation length for a chain composed of patches described by the matrix $M_{g,c=0}$ is given by

$$\xi_g = \frac{g}{\ln(\lambda_g^+/\lambda_g^-)} = -\frac{g}{\ln(C_g(g;T_{c,g}))}. \qquad (8)$$

Based on the above expressions for thermodynamical properties, we obtain analytical expressions for $\lambda_{g,c}^+$

and $\lambda_{g,c}^{-}$ given by:

$$\lambda_{g,c}^{\pm} = \sum_{j=1}^{2^{g-1}} [a_g(j,1) \pm a_g(2^g+1-j,1)]\gamma_{g,c}^{\pm}(j), \qquad (9)$$

where

$$\begin{cases} \gamma_{g,c+1}^{\pm} = a_g(2j-1,2)\gamma_{g,c}^{\pm}(2j-1) + a_g(2j,2)\gamma_{g,c}^{\pm}(2j) \\ \qquad\qquad\qquad\qquad\qquad j = 1,2,\ldots,2^{g-2} \\ \gamma_{g,c+1}^{\pm} = \pm a_g(2j-1,2)\gamma_{g,c}^{\pm}(2^g+2-2j) \\ \quad \pm a_g(2j,2)\gamma_{g,c}^{\pm}(2^g+1-2j), \; j = 2^{g-2}+1\ldots 2^{g-1}, \\ a_g(j,1) = (\prod_{i=1}^{g} P_i)_{1,j}, \qquad\quad j = 1,2\ldots,2^{g-1} \\ a_g(j,2) = (P_j)_{1,j}, \qquad\qquad j = 1,2\ldots,2^{g-1} \end{cases} \qquad (10)$$

with $\gamma_{g,c=0}^{\pm}(j) = \pm 1$.

Since the eigenvalues $\lambda_{g,c}^{\pm}$ are given by (9), we can easily derive expressions for their derivatives with respect to T and h, from which the thermodynamic functions, expressed in terms of the derivatives of the free energy, can be easily evaluated without resorting to numerical differentiation.

To obtain the scaling properties of ξ we will make use of the FRS framework [7]. This scheme proposes a scaling hypothesis which is formally similar to the well known finite size scaling, which compares the behaviors of finite size systems with different number of components. In FRS, one assumes similar relations among systems with distinct number of coupling constants, hence of different interaction ranges. Starting from the assumption that, close to the critical temperature $\overline{T}$, any thermodynamical function $y(t)$ of the reduced temperature $t = (T - \overline{T}/\overline{T})$ is described by a power law

$$y(t) = A_0 t^{-\rho}, \qquad (11)$$

and that a finite number g of coupling constants modifies the actual criticality by a correction factor f, it is proposed that

$$y_g(t) = y_\infty(t) f(g/\xi_\infty). \qquad (12)$$

Using (11) and (12), it is possible to show that, close to $\overline{T}$, the condition

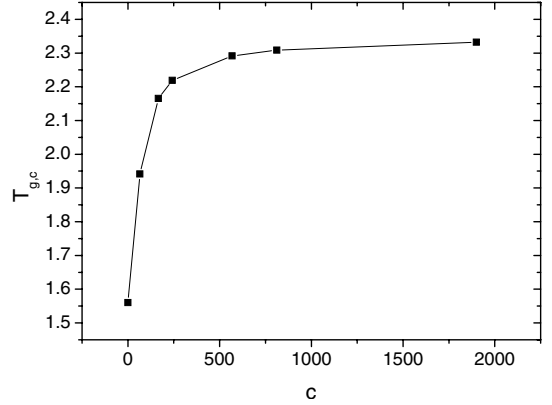$$\frac{\xi_g(t)}{\xi_{g+1}(t')} = \frac{g}{g+1}, \qquad (13)$$

holds. Or, equivalently, t and t' are related by

$$t' = \left[\frac{g}{g+1}\right]^{\frac{1}{\nu}} t. \qquad (14)$$

The critical temperature $\overline{T}_g$ at order g is obtained by the condition

$$\frac{\xi_g(\overline{t}_g)}{\xi_{g+1}(\overline{t}_g)} = \frac{g}{g+1}, \qquad (15)$$

and it is expected that the series of values $\overline{t}_g$ converge to the actual $\overline{t} = 0$ in the limit $g \to \infty$.

Fig. 2. Dependence of $\overline{T}_{g,c}$ on c, for $\alpha = 1.8$ and a fixed value of g, for example, $g = 10$.

Linearizing and expanding around $\overline{T}_g$, leads to

$$\frac{\xi_{g+1}(t)}{\xi_g(t')} = \frac{\xi_{g+1}(\overline{t}_g) + (d\xi_{g+1}(\overline{t}_g)/dt)t}{\xi_g(\overline{t}_g) + (d\xi_g(\overline{t}_g)/dt)t'}. \qquad (16)$$

Combining (14) with (16), we obtain estimates of the critical exponent ν_g as

$$\frac{1}{\nu_g} = \frac{\ln\left(\frac{d\xi_g(\overline{t}_g)/dt}{d\xi_{g+1}(\overline{t}_g)/dt}\right) - 1}{\ln\left(\frac{g}{g+1}\right)}. \qquad (17)$$

3 Results

In reference [21], we have obtained a series of estimates $\overline{T}$ for the critical temperature, based on the observed behavior for $C_g(r;T)$, which corresponds exactly to the condition expressed by (15). In such case, we have considered $N = g + 1$, i.e., $c = 0$. The actual value of $\overline{T}$ has been evaluated by an extrapolation procedure, with very good accuracy, although the individual values $\overline{T}_g$ are not so close to $\overline{T}$.

To address the question of the critical behavior near the critical temperature, we need to be in a much closer neighborhood of $\overline{T}$. So we found it becomes necessary to consider a larger number of spins in the chain, what requires larger values of g and c. Increasing g leads to exponential growth of storage capacity and computing time. We found, however, that taking a better approximation to the infinite chain, what amounts to increasing c for a fixed value of g, makes it possible to reach a closer neighborhood of $\overline{T}$.

To this purpose, we evaluate a series of critical temperatures $\overline{T}_{g,c}$ depending on both g and c, from which we obtain first $\overline{T}_g = \lim_{c\to\infty} \overline{T}_{g,c}$ and $\overline{T} = \lim_{g\to\infty} \overline{T}_g$. Each $\overline{T}_{g,c}$ is obtained by imposing the condition (16), where each ξ_g actually indicates $\xi_{g,c}$, which is evaluated for corresponding values of g and c with the help of (2). In Figure 2, we show how $\overline{T}_{g,c}$ depends on c, for a fixed value of g when $\alpha = 1.8$; the behavior does not change for other values of g.

Table 1. Values of $\overline{T_g}$ and ν_g, for odd values of g and $\alpha = 1.2$. Data are displayed for accuracy $\epsilon = 10^{-4}$ and $\epsilon = 10^{-3}$ that was required for the limiting value of $\overline{T_g}$. The corresponding smallest value of c at which the accuracy was reached is also indicated.

$\alpha = 1.2$		$\epsilon = 10^{-4}$			$\epsilon = 10^{-3}$	
g	$c(\epsilon)$	$\overline{T_{g,c}}$	$\nu_{g,c}$	$c(\epsilon)$	$\overline{T_{g,c}}$	$\nu_{g,c}$
3	684	5.5419339	2.4783289	208	5.3921944	2.4336811
5	1058	6.6798492	2.7505706	314	6.4445671	2.6616931
7	1396	7.3835491	2.9643628	410	7.0728148	2.8315852
9	1706	7.8632795	3.1338069	492	7.4796335	2.9552605
11	1990	8.2103090	3.2697569	566	7.7599961	3.0455192
13	2260	8.4726154	3.3805739	632	7.9590277	3.1099467
15	2500	8.6753271	3.4710018	692	8.1041143	3.1550949
VBS extrapolation	-	9.482310	3.803537	-	8.910613	3.288895

In our numerical evaluation, $\overline{T_g}$ has been obtained with a given accuracy, as we increase the value of c until $\mid \overline{T_{g,c+1}} - \overline{T_{g,c}} \mid < \epsilon$, where the highest accuracy corresponds to $\epsilon \simeq 10^{-4}$. For such situation, c can be as large as 2500, depending on the values of g and α. The largest value of $g = 18$ has been obtained for $\alpha = 1.6$.

In Tables 1–4, we show the values of $\overline{T_g}$ and ν_g, for distinct values of g and $1 < \alpha < 2$: $\alpha = 1.2, 1.4, 1.6$ and 1.8, for a fixed value of accuracy $\epsilon = 10^{-4}$. In order to analyse the effect of the accuracy ϵ, for values of α, inside and outside the mean field region, $\alpha = 1.2$ (Tab. 1) and 1.8 (Tab. 4) respectively, we include, in these tables, the data for another value of $\epsilon = 10^{-3}$. It is easy to see that the value of c is very sensible to the accuracy ϵ; in Figure 2 we plot the critical temperature $\overline{T_{g,c}}$ for increasing values of c, corresponding to, respectively, decreasing values of ϵ.

Regarding the values of $\overline{T_{g,c}}$ and $\nu_{g,c}$, we see that they form a monotonic series, increasing with respect to both g and c, approaching $\overline{T}$ from the lower side. In order to extrapolate the finite series results for a limit value $\overline{T}$ and ν, we have used the Vanden Broeck and Schwartz (VBS) extrapolation procedure used both in [7] and [21]. We observe that the values of $\overline{T_{g,c}}$, for any value of α, are much larger than those obtained previously for $c = 0$ [21]. But, despite of this, some of the extrapolated values are smaller than those predicted from the series $\overline{T_{g,c=0}}$. This is certainly due to the fact that, as $c(\epsilon)$ depends on g, the chain has different numbers of spins (see Tabs. 1–4).

In Table 5, we show the VBS extrapolated values of ν for the same values of α and a fixed value of $\epsilon = 10^{-4}$. Here we are lead to the most interesting result of this paper as it represents an improvement to previous results. For the purpose of comparison, we use the known exact values for $\nu = 1/(\alpha-1)$ in the range $1 < \alpha \leq 1.5$, when the values obtained from the mean field analysis should prevail also for those from an exact solution. It is clear that our VBS extrapolation leads to a better agreement with the exact value in comparison to the corresponding VBS extrapolated results in [7]. Also, based on the same comparison, we see, in Table 1, that the extrapolated values of $\nu = \lim_{g \to \infty} \nu_g$ get better when ϵ is reduced from 10^{-3} to 10^{-4}. Note that we use larger values of N ($N \approx 15$ for all values of α) than those ones used in reference [7], what

Table 2. Values of $\overline{T_g}$ and ν_g, for odd values of g and $\alpha = 1.4$. Data are displayed for accuracy $\epsilon = 10^{-4}$ that was required for the limiting value of $\overline{T_g}$ and the corresponding smallest value of c is also indicated.

$\alpha = 1.4$		$\epsilon = 10^{-4}$	
g	$c(\epsilon)$	$\overline{T_{g,c}}$	$\nu_{g,c}$
3	492	3.8408632	2.1731057
5	694	4.3296422	2.2598101
7	856	4.5894104	2.3162027
9	994	4.7474461	2.3539294
11	1112	4.8515617	2.3793714
13	1216	4.9241076	2.3964600
15	1310	4.9767888	2.4076891
VBS extrapolation	-	5.022703	2.437089

Table 3. Values of $\overline{T_g}$ and ν_g, for odd values of g and $\alpha = 1.6$. Data are displayed for accuracy $\epsilon = 10^{-4}$ that was required for the limiting value of $\overline{T_g}$ and the corresponding smallest value of c is also indicated.

$\alpha = 1.6$		$\epsilon = 10^{-4}$	
g	$c(\epsilon)$	$\overline{T_{g,c}}$	$\nu_{g,c}$
3	382	2.8178502	2.0697087
5	510	3.0448535	2.0947359
7	608	3.1555507	2.1052548
9	688	3.2190642	2.1091374
11	756	3.2591533	2.1095755
13	814	3.2859512	2.1080216
15	866	3.3047686	2.1053158
17	910	3.3180850	-
18	932	3.3234981	-
VBS extrapolation	-	3.328188	2.109820

should be very important for the estimation of the the critical exponents. In this sense, the efficient TM method that we have applied is very useful to reduce the CPU time of the numerical calculations.

The values of ν_g for $c = 0$ are indeed far away from the exact ones. This indicates that, although the value of $\overline{T}$ can be evaluated with good precision from the $c = 0$ finite

Table 4. Values of $\overline{T_g}$ and ν_g, for odd values of g and $\alpha = 1.8$. Data are displayed for accuracy $\epsilon = 10^{-4}$ and $\epsilon = 10^{-3}$ that was required for the limiting value of $\overline{T_g}$ and the corresponding smallest value of c is also indicated.

$\alpha = 1.8$		$\epsilon = 10^{-4}$			$\epsilon = 10^{-3}$	
g	$c(\epsilon)$	$\overline{T_{g,c}}$	$\nu_{g,c}$	$c(\epsilon)$	$\overline{T_{g,c}}$	$\nu_{g,c}$
3	314	2.1323856	2.1231430	94	2.0633542	2.1560530
5	408	2.2243788	2.1639020	122	2.1346603	2.2165373
7	482	2.2642364	2.1873097	142	2.1575740	2.2647878
9	540	2.2847804	2.2036550	158	2.1642335	2.3109579
11	592	2.2967828	2.2163352	172	2.1644328	2.3583671
13	636	2.3039057	2.2273095	186	2.1629377	2.4063897
15	676	2.3083424	2.2372586	198	2.1588773	2.4598516
VBS extrapolation	-	2.315334	2.246300	-	2.167883	2.910225

Table 5. Comparative table of VBS extrapolated values of $\overline{T}$ and ν, for distinct values of $1 < \alpha < 2$, using $\epsilon = 10^{-4}$, obtained from the data in Tables 1–4. We also include the corresponding value ν, calculated by Glumac and Uzelac [7], ν_{gu}, and the exact value of ν, ν_{ex} for the mean field values of α.

α	ν	ν_{gu}	ν_{ex}
1.2	3.803537	7.0	5.0
1.4	2.437089	2.7	2.5
1.6	2.109820	2.15	-
1.8	2.246300	2.22	-

size data, the critical properties require indeed to probe the system in a close neighborhood of $\overline{T}$. So, it is amazing to observe that our TM approach was able to be adapted to perform this most sensitive task.

4 Conclusions

In this work we presented results for the critical exponent ν for the long-range Ising chain, based on the numerical evaluation of the eigenvalues of a 2×2 TM, that condenses the information regarding the Boltzmann weights for all configurations of a finite size chain of $g + c + 1$ spins including interaction among spins up to g sites apart. The adopted approach requires the minimum possible storage space and avoids the necessity of eigenvalue evaluation. The results were obtained with a double precision Fortran code implemented on a common desk computer.

The comparison of our estimates with similar results reported by other authors shows that they are of the same quality or better than those obtained by a TM procedures that requires much larger matrices. In particular, we have used the known exact values of ν in the range $1 < \alpha \le 1.5$ to check the validity of our results.

As compared to our previous implementation of our TM framework, we have shown that it is indeed reliable and that it could be successfully extended to much larger values of c, what was required in order to probe the close neighborhood of $\overline{T}$. Despite the fact that analytical expression for the derivatives of the eigenvalues represents an important achievement of our approach and avoids the numerical differentiation of ξ, we have observed that numerical overflows in our Fortran code did not allow push-

ing the series of ν_g to the same larger values of $g = 24$ that were reached in our first evaluation of the critical temperature when $c = 0$. Efforts to sidestep this effect and to include the evaluation of the magnetization exponent β are currently being undertaken to present a more complete analysis of this model.

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